

Mathematical Proofs

Announcements

- ***Pset 0***

- Due Monday (but try to finish by Friday)

- ***Pset 1***

- Goes out Friday, due following Friday
- LaTeX Beginner's Quick Start Tutorial (LaTeX is the preferred tool for writing homework in this class)

- ***Office Hours***

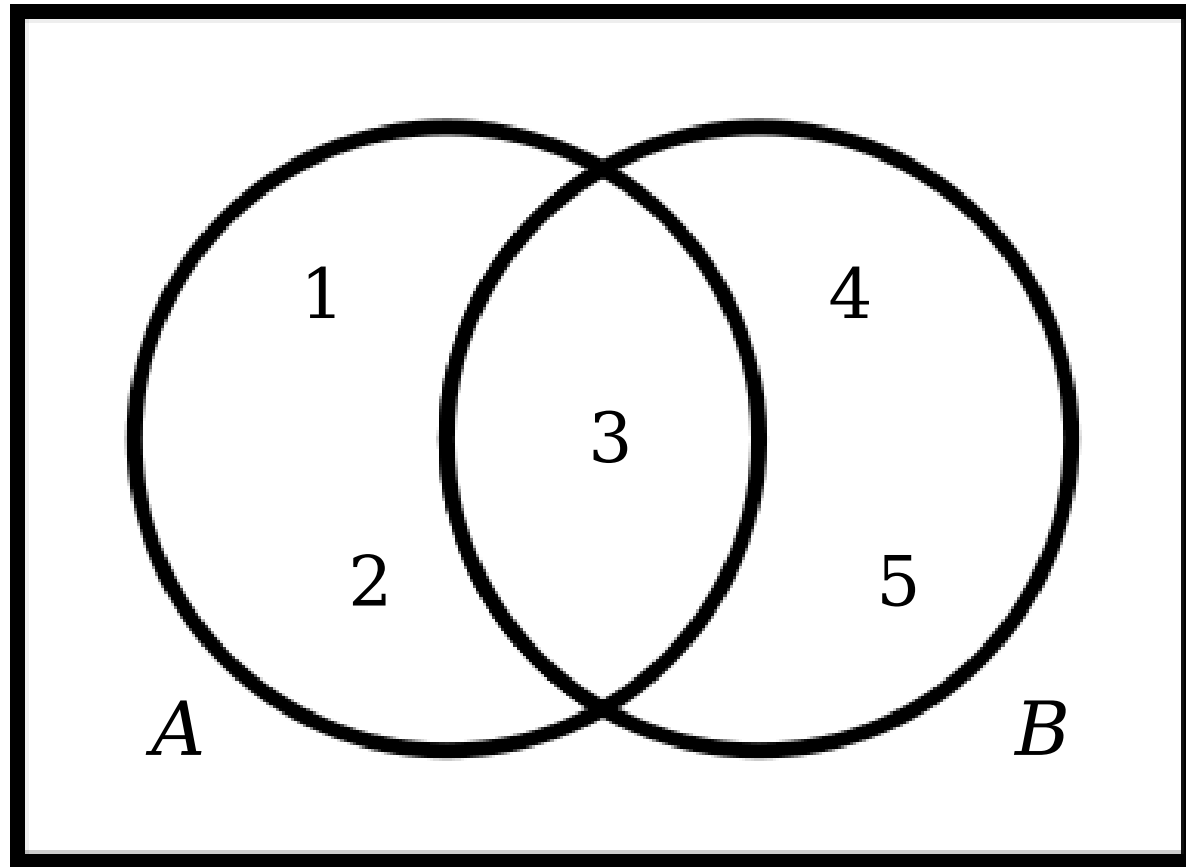
- They start Monday! Schedule will be on the course website by Friday. They will be accessible in person or by Zoom.

Outline for Today

- ***How to Write a Proof***
 - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
 - Working with odd and even numbers.
- ***Universal and Existential Statements***
 - Two important classes of statements.
- ***Variable Ownership***
 - Who owns what?

Combining Sets

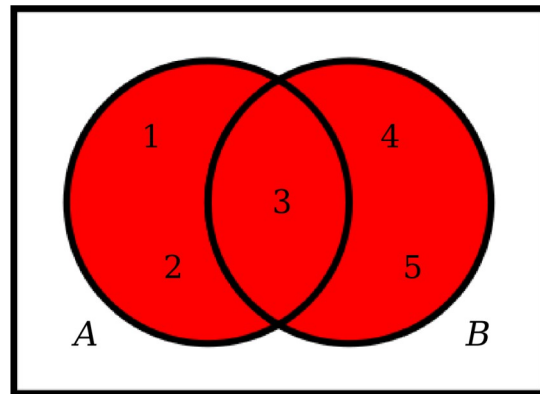
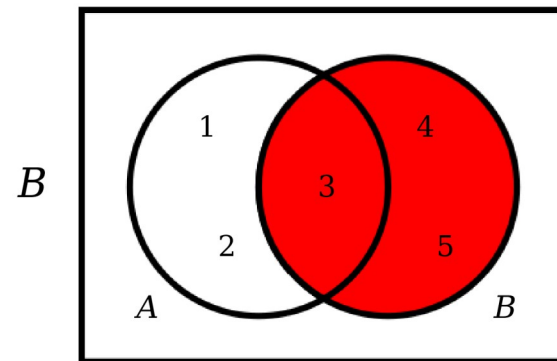
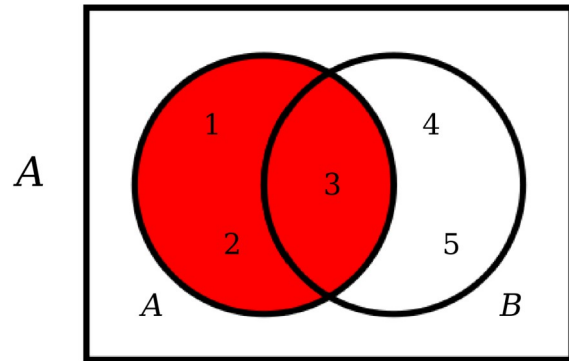
Venn Diagrams



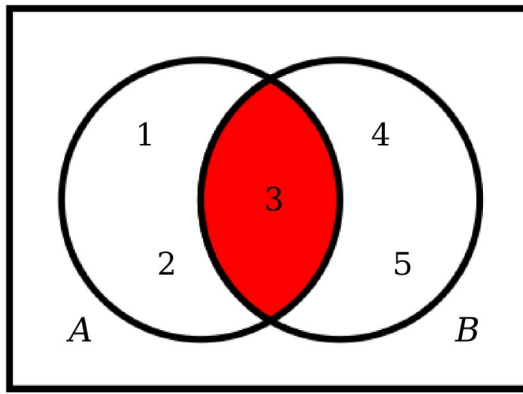
$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



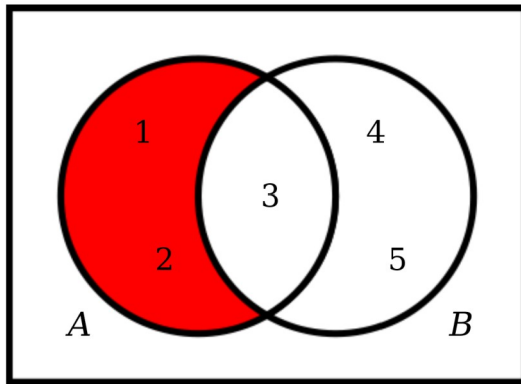
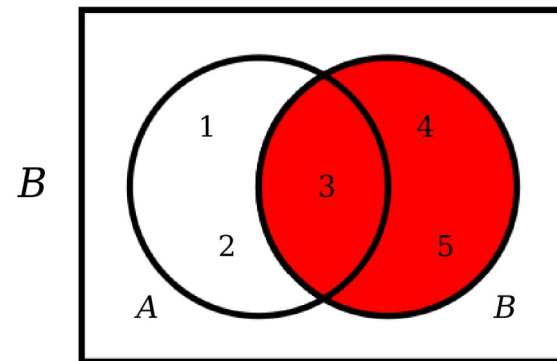
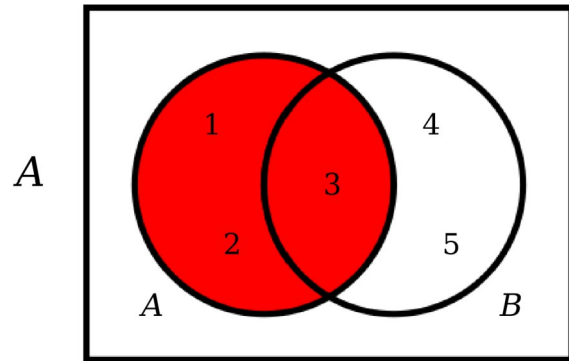
Union
 $A \cup B$
 $\{ 1, 2, 3, 4, 5 \}$



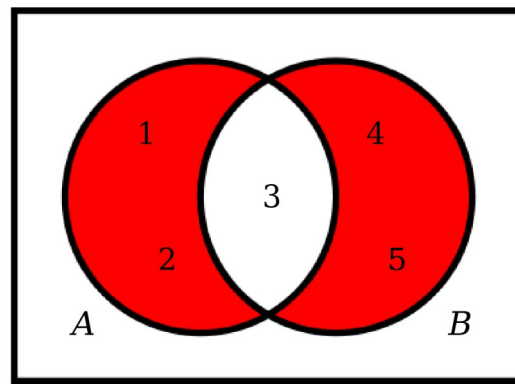
Intersection
 $A \cap B$
 $\{ 3 \}$

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



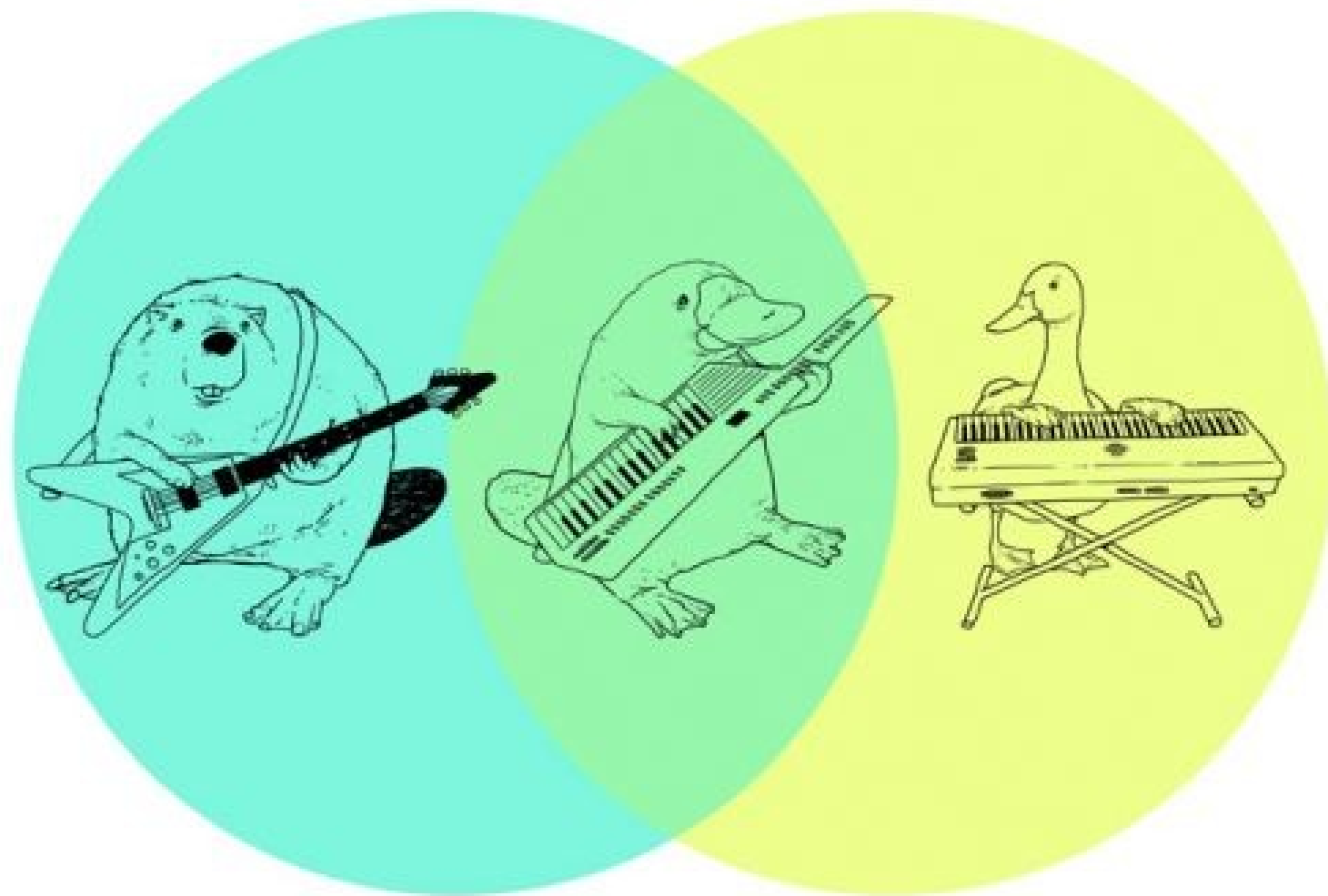
Difference
 $A - B$
 $A \setminus B$
 $\{ 1, 2 \}$



Symmetric
Difference
 $A \Delta B$
 $\{ 1, 2, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



What is a Proof?

A *proof* is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.

Proofwriting is not like other forms of writing, or even other forms of math problems, and we understand this can be a big adjustment! Here is some advice from years of teaching proofwriting.

Rule: Proofs are meant to argue something precisely and completely.

Advice: Well trained readers should find this persuasive, however *precision* and *completeness* are the goals you should have in mind, not *persuasion*.

Action: Try to detach yourself from the notion of “persuasive” in the usual social-emotional-rhetorical way that we think of persuasion.

Rule: Skipping even one step of a proof is a big deal—it makes the proof logically invalid.

Advice: Think of a proof as written driving directions from Point A to Point B, for someone who has never been, and who doesn't have a GPS/phone. If you leave out one left turn onto Main St., the driver will never get to the next right turn onto Maple St. They simply can't continue, and will be *permanently lost!* Having "some of the right intuition" isn't enough.

Action: Try not to hate us when we grade in a way that might seem "nit-picky." Meticulous attention to detail is important to learning the proofwriting skill.

Writing our First Proof

Definitions:

An integer n is called ***even*** if there is an integer k where $n = 2k$.

An integer n is called ***odd*** if there is an integer k where $n = 2k + 1$.

Theorem: For all integers n , if n is even, then n^2 is even.

Theorem: For all integers n , if n is **even**, then n^2 is **even**.

Step 1: find the formal definitions for any terms in the theorem. (Not just what you intuitively understand the concept of “even” to mean.)

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“For all” means we need this to be true for all examples of elements of the set integers. So we challenge our reader: “pick ANY integer!!”

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The “if...then...” means that the “then” part only holds when the “if” part is true. So we are going to tell our reader to pick any integer **assuming** that the “if” condition is true of it. In other words, only pick even integers.

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Our First Proof!

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Since n is even, there is some integer k such that $n = 2k$.

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Let's Try Some Examples!

$$2^2 = 4 = 2 \cdot \mathbf{2}$$

$$10^2 = 100 = 2 \cdot \mathbf{50}$$

$$0^2 = 0 = 2 \cdot \mathbf{0}$$

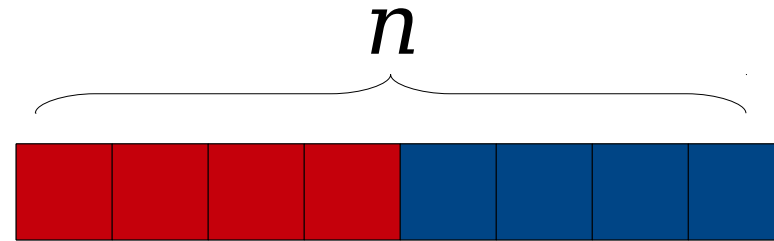
$$(-8)^2 = 64 = 2 \cdot \mathbf{32}$$

$$n^2 = 2 \cdot \mathbf{?}$$

What's the pattern? How do we predict this?

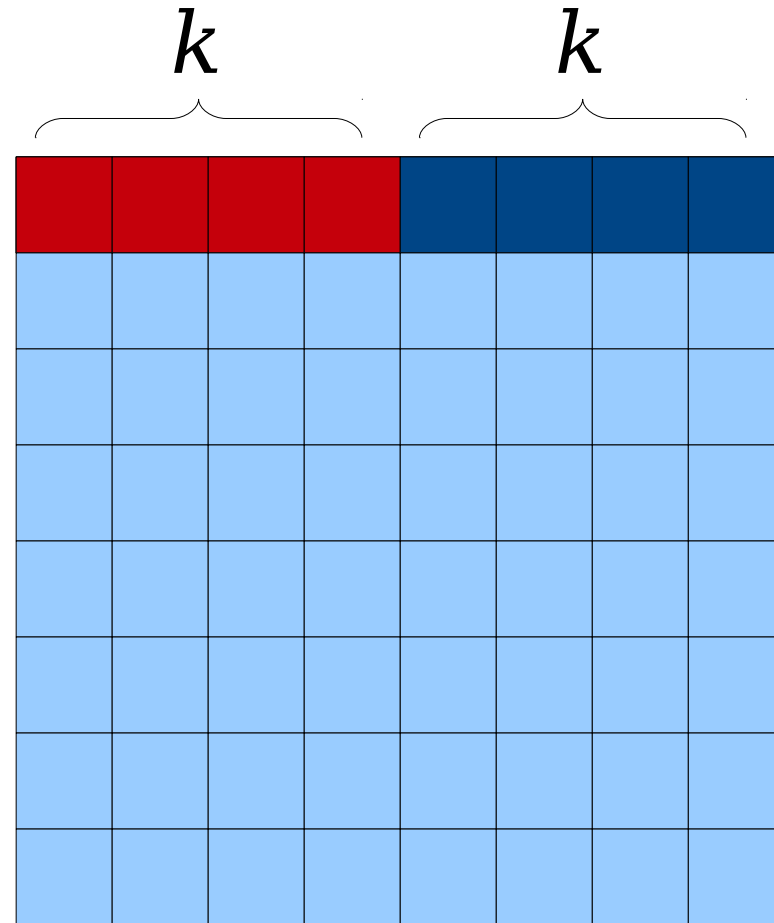
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Let's Draw Some Pictures!



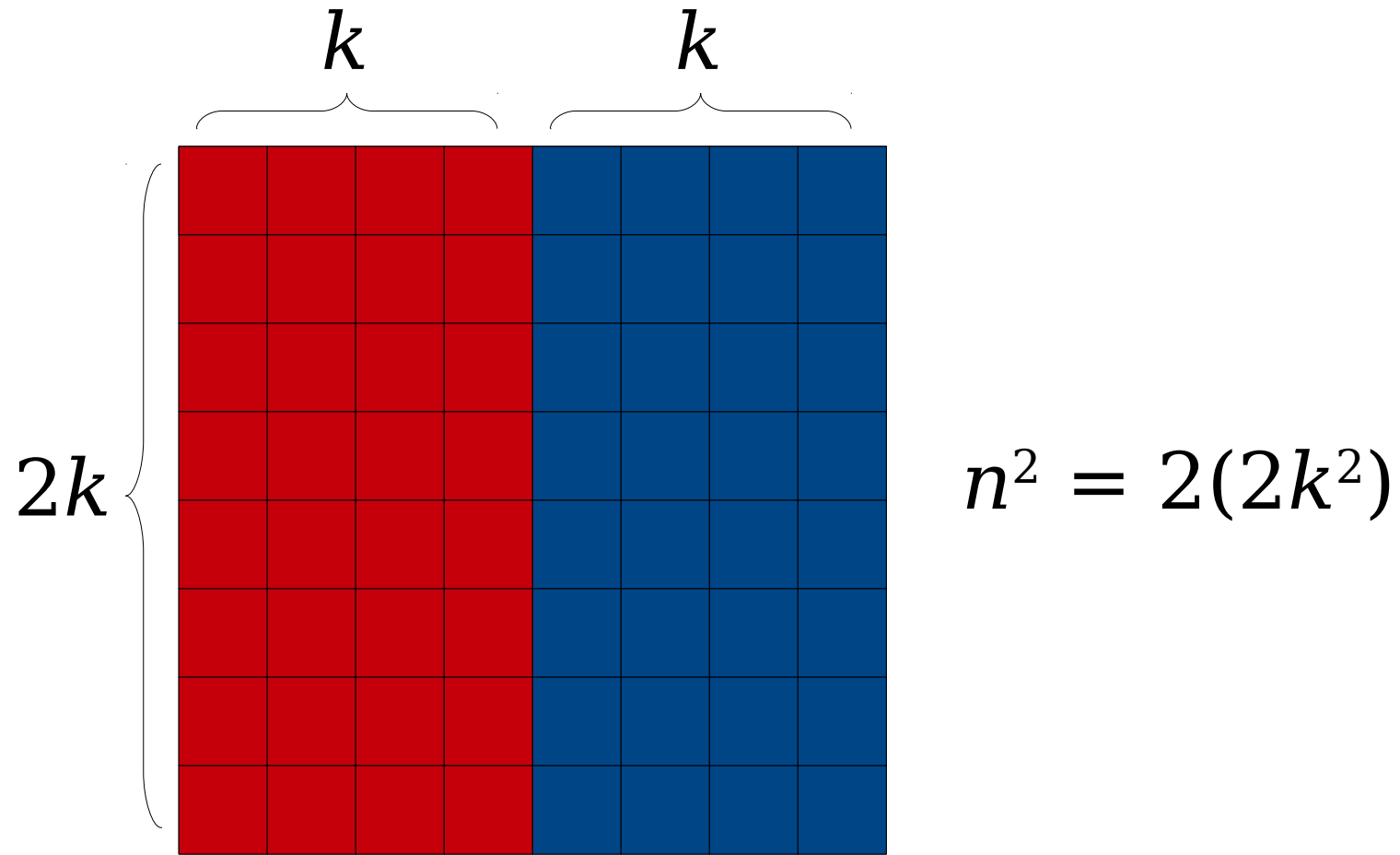
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Since n is even, there is some integer k such that $n = 2k$. This means that

$$n^2 = (2k)^2$$

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$$\begin{aligned}n^2 &= (2k)^2 \\ &= 4k^2\end{aligned}$$

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Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned}n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2).\end{aligned}$$

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This symbol means “end of proof.” It’s basically math nerd “mic drop.”

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Our First Proof!

Theorem: For all integers n , if n is even, then n^2 is even.

Proof: Pick an arbitrary even integer n . We want to show that n^2 is even.

Since n is even, we know that $n = 2k$. Then

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

From this, we see that n^2 is even (namely, $2k^2$) which is even, which is

To prove a statement of the form

“If P is true, then Q is true,”

start by assuming that P is true. Here, we’re inviting the reader to pick their favorite even integer.

Our First Proof!

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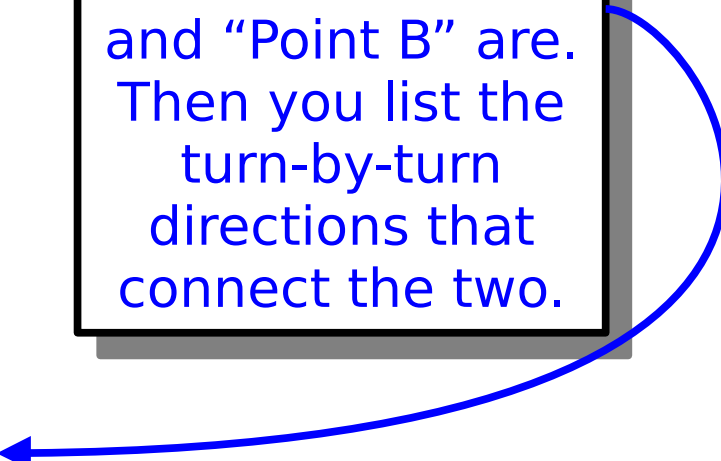
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start by assuming that P is true. Here, we’re inviting the reader to pick their favorite even integer.

After assuming P is true, you need to show that Q is true. Here, we’re telling the reader where we’re headed.

This is basically starting your driving directions by announcing what “Point A” and “Point B” are. Then you list the turn-by-turn directions that connect the two.



Our First Proof!

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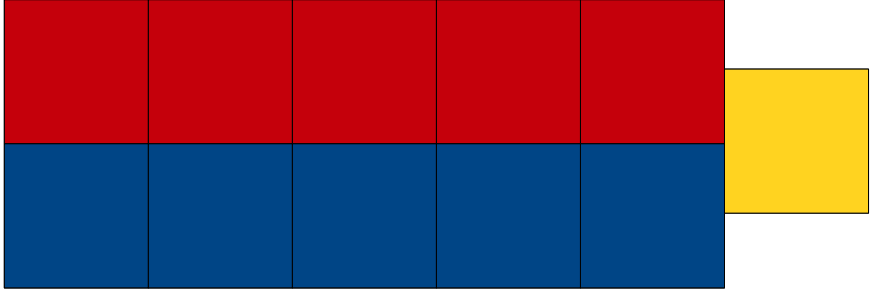
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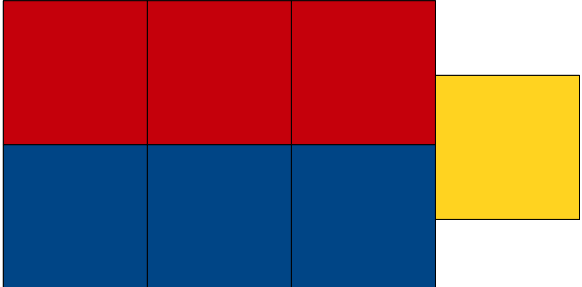
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Our Next Proof

Theorem: For any integers m and n ,
if m and n are odd, then $m + n$ is even.

Theorem: For any integers m and n ,
if m and n are **odd**, then $m + n$ is even.

11  $2 \cdot 5 + 1$

7  $2 \cdot 3 + 1$

1  $2 \cdot 0 + 1$

An integer n is called **odd** if there is an integer k where $n = 2k + 1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: For any integers m and n ,
if m and n are odd, then $m + n$ is even.

Let's Try Some Examples!

$$1 + 1 = 2 = 2 \cdot \mathbf{1}$$

$$137 + 103 = 240 = 2 \cdot \mathbf{120}$$

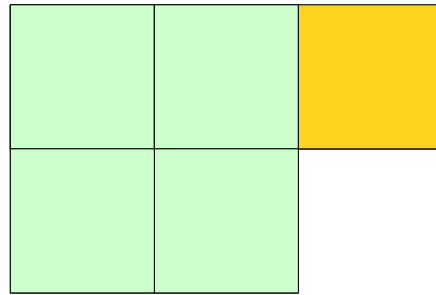
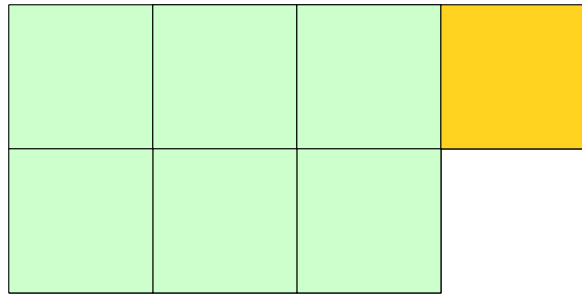
$$-5 + 5 = 0 = 2 \cdot \mathbf{0}$$

$$m + n = 2 \cdot \mathbf{?}$$

What's the pattern? How do we predict this?

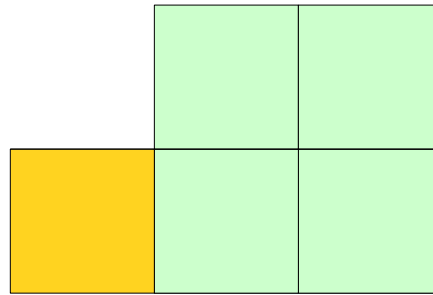
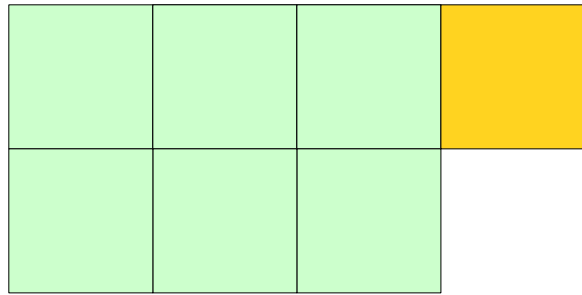
Theorem: For any integers m and n , if m and n are odd, then $m+n$ is even.

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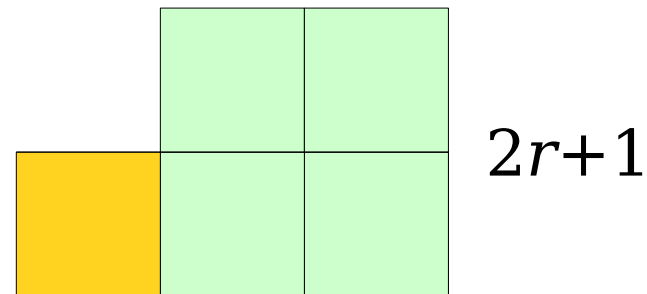
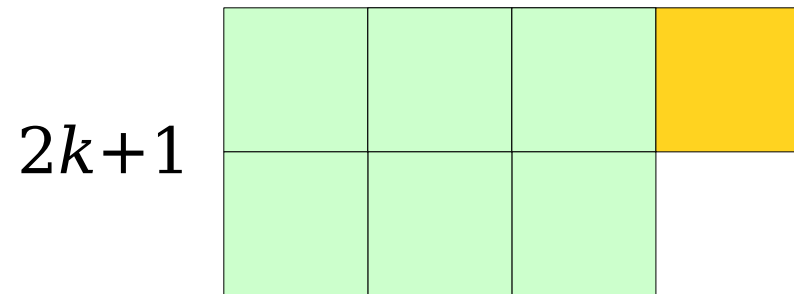
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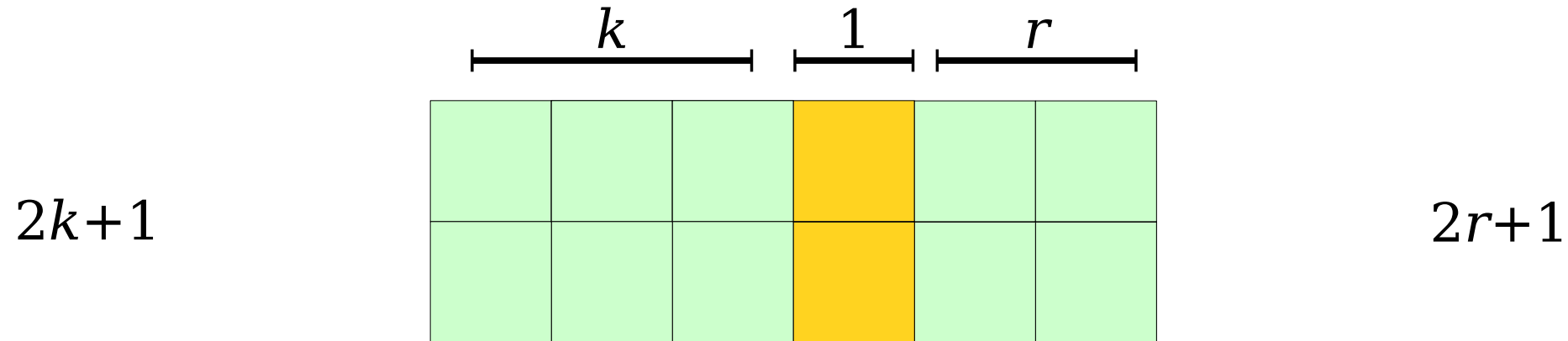
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Let's Do Some Math!



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Let's Do Some Math!



$$(2k+1) + (2r+1) = 2(k + r + 1)$$

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Proof: Consider any arbitrary integers m and n where m and n are odd. We want to show that

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Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \quad (1)$$

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Proof: Consider any arbitrary integers m and n where m and n are odd. We want to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

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Similarly, because n is odd there must be some integer r such that

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$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \end{aligned}$$

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Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$.

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Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required.

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Similarly, because

By adding equation

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We ask the reader to make an **arbitrary choice**. Rather than specifying what m and n are, we're signaling to the reader that they could, in principle, supply any choices of m and n that they'd like.

By letting the reader pick m and n arbitrarily, anything we prove about m and n will generalize to all possible choices for those values.

Theorem: For any integers m and n , if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We want to show that $m + n$ is even.

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To prove a statement of the form

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Similarly, b

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By adding e

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Similarly, we can show that $n = 2r + 1$ for some integer r . We wish to show that

after assuming P is true, you need to show that Q is true.

By adding

$$= 2k + 2r + 2$$

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Theorem: For any integers m and n , if m and n are odd, then $m + n$ is even.

Proof: Consider any odd. We want to show that $m + n$ is even. Since m is odd, we

Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

$$m = 2k + 1. \quad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \quad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \quad (3)$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n , if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We want to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \quad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \quad (2)$$

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

we can learn that

$$m + n = 2k + 1 + 2r + 1$$

$$= 2k + 2r + 2$$

$$= 2(k + r + 1). \quad (3)$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

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Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
 - **Theorem:** The sum and difference of any two even numbers is even.
 - **Theorem:** The sum and difference of an odd number and an even number is odd.
 - **Theorem:** The product of any integer and an even number is even.
 - **Theorem:** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For all odd integers n ,
there exist integers r and s where $r^2 - s^2 = n$.

Theorem: For all odd integers n ,
there exist integers r and s where $r^2 - s^2 = n$.

This result is true for every possible
choice of odd integer n . It'll work for $n =$
 $1, n = 137, n = 103$, etc.

Theorem: For all odd integers n ,
there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s . Rather, we're saying that **somewhere out there** are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

For all x , [some-property] holds for x .

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

There is some x where [some-property] holds for x .

- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

There is an x where [some-property] holds for x .

- ***Approach:*** Search far and wide, find a concrete example value for x that has the right property. In the proof, (1) announce the find to your reader, then (2) show why your choice is correct.

Let's Try Some Examples!

$$1 = \underline{\quad}^2 - \underline{\quad}^2$$

$$3 = \underline{\quad}^2 - \underline{\quad}^2$$

$$5 = \underline{\quad}^2 - \underline{\quad}^2$$

$$7 = \underline{\quad}^2 - \underline{\quad}^2$$

$$9 = \underline{\quad}^2 - \underline{\quad}^2$$

Theorem: For all odd integers n ,
there exist integers r and s where $r^2 - s^2 = n$.

Let's Try Some Examples!

$$1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = \mathbf{3}^2 - \mathbf{2}^2$$

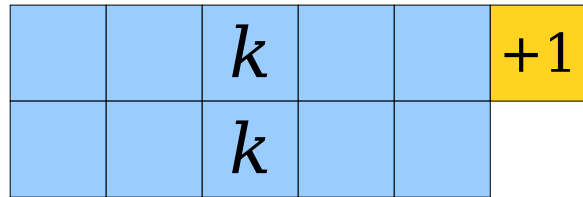
$$7 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = \mathbf{5}^2 - \mathbf{4}^2$$

We've got a pattern - but why does this work?

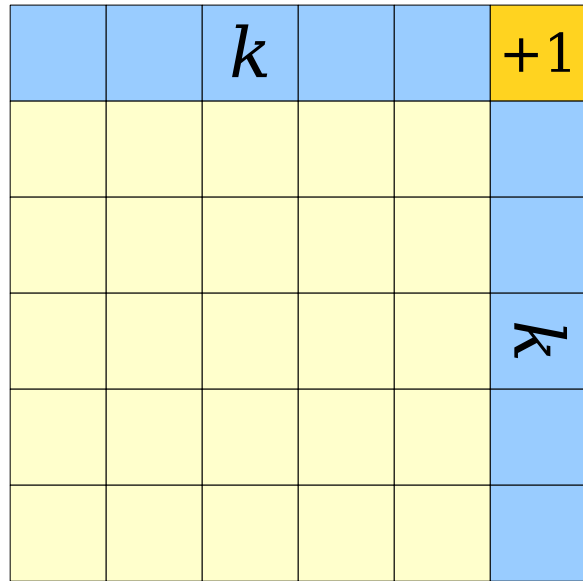
Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

Let's Draw Some Pictures!



Theorem: For all odd integers n ,
there exist integers r and s where $r^2 - s^2 = n$.

Let's Draw Some Pictures!



$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For all odd integers n ,
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Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

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Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n . We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 - k^2$$

Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

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$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \end{aligned}$$

Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

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This means that $r^2 - s^2 = n$, which is what we needed to show.

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Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n . We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd,
 $n = 2k + 1$
that

We ask the reader to make an **arbitrary choice**. Rather than specifying what n is, we're signaling to the reader that they could, in principle, supply any choice n that they'd like.

there
we see

$$r = k + 1$$

$$s = k$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n . We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know that $n = 2k + 1$. Now, let $r = k + 1$ and $s = k$. We will demonstrate that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For all odd integers n , there exist integers r and s where $r^2 - s^2 = n$.

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This means that $r^2 - s^2 = n$, which is what we wanted to show. ■

We're trying to prove an existential statement. First we announce concrete choices of the objects being sought out.

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Proof: Pick an arbitrary odd integer n . We will show that there exist integers r and s where $r^2 - s^2 = n$.

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This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: If n is an integer,
then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Floors and Ceilings

- The notation $\lceil x \rceil$ represents the **ceiling** of x , the smallest integer greater than or equal to x .

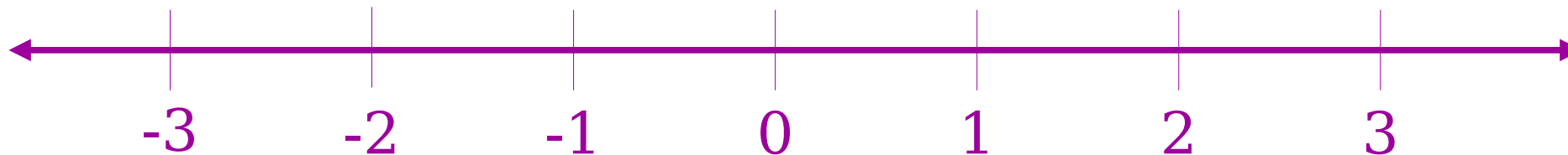
$$\lceil 1 \rceil = 1 \quad \lceil 1.5 \rceil = 2$$

$$\lceil -1 \rceil = -1 \quad \lceil -1.5 \rceil = -1$$

- The notation $\lfloor x \rfloor$ represents is the **floor** of x , the largest integer less than or equal to x .

$$\lfloor 1 \rfloor = 1 \quad \lfloor 1.5 \rfloor = 1$$

$$\lfloor -1 \rfloor = -1 \quad \lfloor -1.5 \rfloor = -2$$



Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Hmmm...are we stuck? Typically, our third sentence (after the “assume” and “WTS”/“want to show” sentences) is an expansion of a definition word, such as “even.” Here we just have an integer, so there’s no definition to expand.

Let’s take a moment to do some work on scratch paper and see if we can find a way forward?

Let's Try Some Examples!

$$\lceil 0/2 \rceil + \lceil 0/2 \rceil = 0 + 0 = 0$$

$$\lceil 1/2 \rceil + \lceil 1/2 \rceil = 1 + 0 = 1$$

$$\lceil 2/2 \rceil + \lceil 2/2 \rceil = 1 + 1 = 2$$

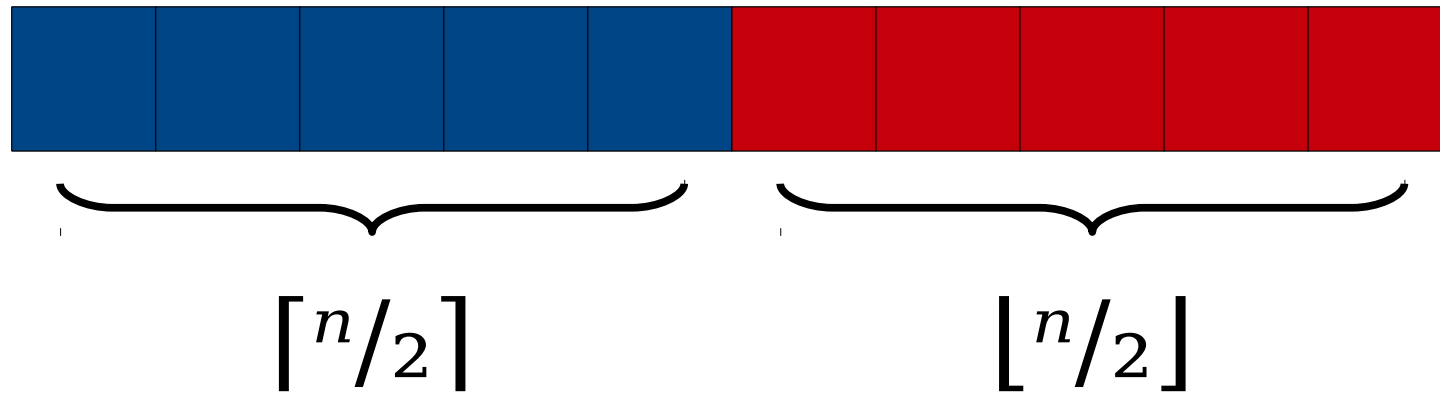
$$\lceil 3/2 \rceil + \lceil 3/2 \rceil = 2 + 1 = 3$$

$$\lceil 4/2 \rceil + \lceil 4/2 \rceil = 2 + 2 = 4$$

Scratch paper work.

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let's Draw Some Pictures!



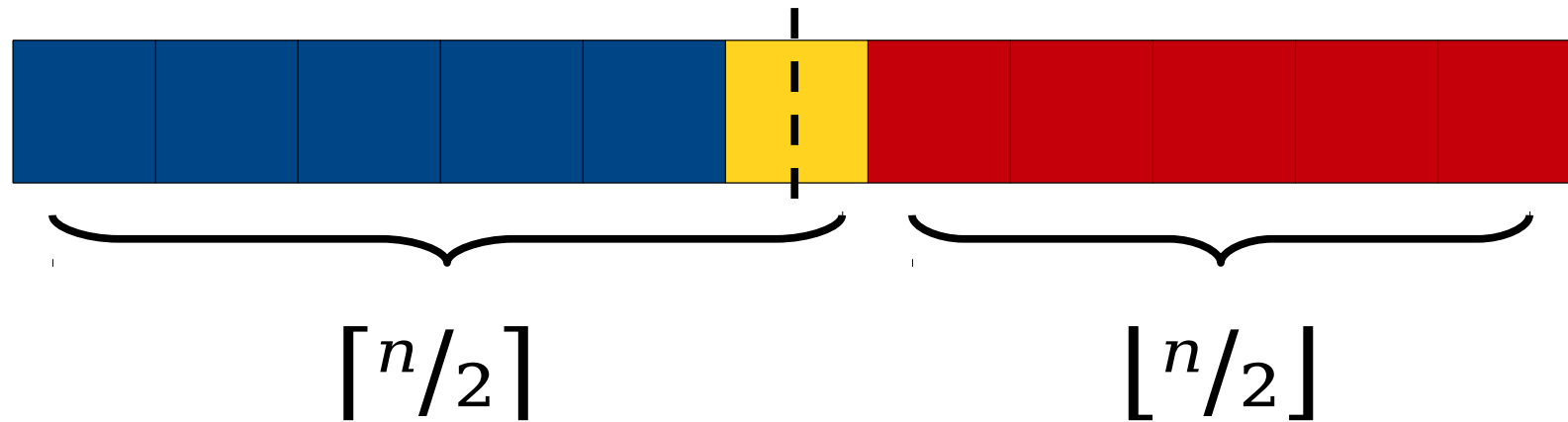
$$n = 2k$$

Scratch paper work.

Hm, too bad we weren't asked to this proof for a theorem that says that n is even—that looks easy! :-(- :-(-

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let's Draw Some Pictures!



$$n = 2k + 1$$

Scratch paper work.

Hm, too bad we weren't asked to this proof for a theorem that says that n is odd—a little harder than even, but we can still find a way. :-(- :-(-

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor = n$.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Turns out that we can just
smash our two proofs
together—the odd one and
the even one—and that
counts as a proof for *all*
integers—yay!!

Thanks, **Proof by Cases!**

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil &= \left\lfloor \frac{2k}{2} \right\rfloor + \left\lceil \frac{2k}{2} \right\rceil \\ &= \lfloor k \rfloor + \lceil k \rceil \\ &= 2k \\ &= n.\end{aligned}$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

The case labels in effect introduce the new assumptions you wish you had, to make the proof solvable. Then you proceed to show your work from there.

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil &= \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lceil \frac{2k+1}{2} \right\rceil \\ &= \left\lfloor k + \frac{1}{2} \right\rfloor + \left\lceil k + \frac{1}{2} \right\rceil \\ &= (k+1) + k \\ &= 2k+1 \\ &= n.\end{aligned}$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

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Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lceil \frac{2k+1}{2} \right\rceil$$

At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n.$$

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Proofs as a Dialog

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Pick an arbitrary odd integer n .

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

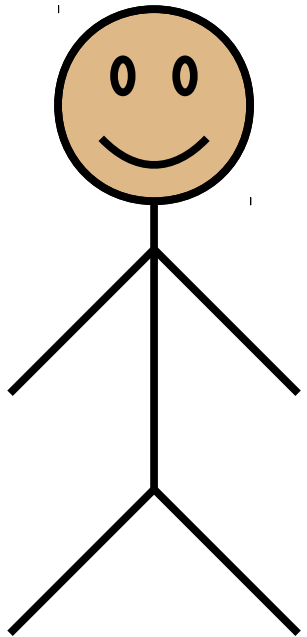
Now, let $z = k - 34$.

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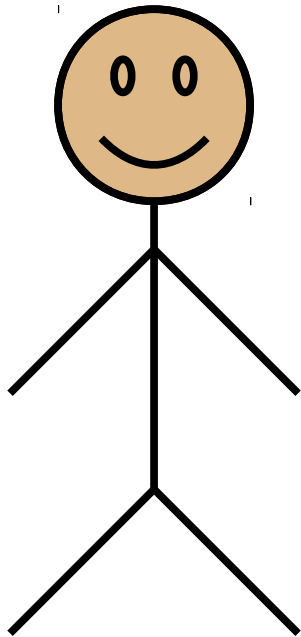
Proof Writer (You)

Proofs as a Dialog

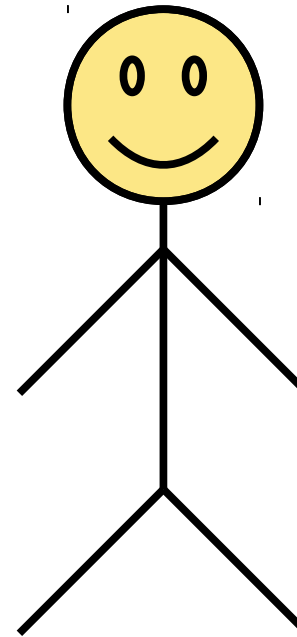
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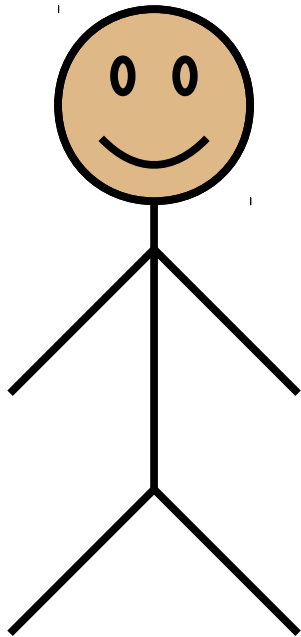
Proof Reader

Proofs as a Dialog

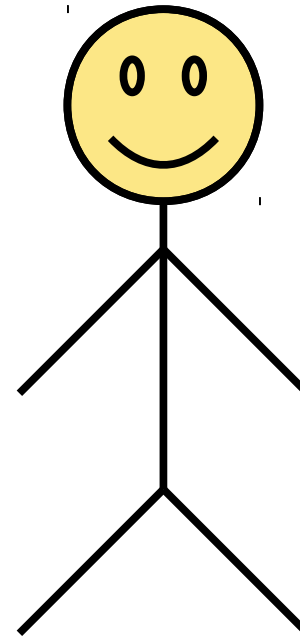
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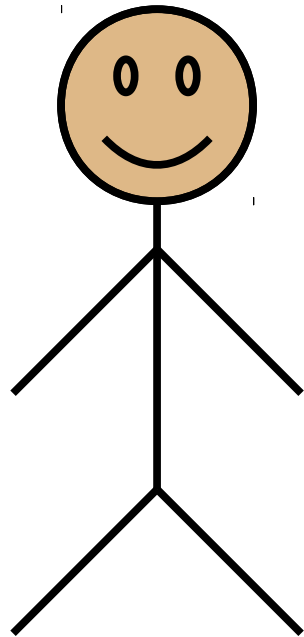
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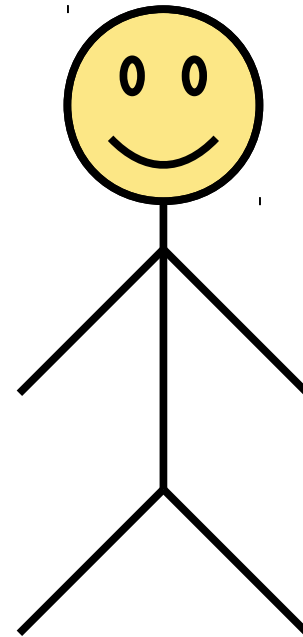
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$$n = 137$$

Reader Picks



Proof Writer (You)



Proof Reader

Proofs as a Dialog

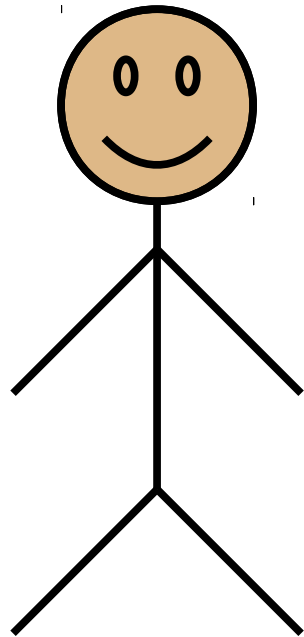
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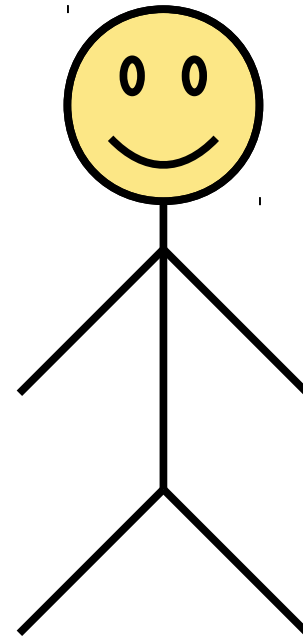
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Proof Writer (You)



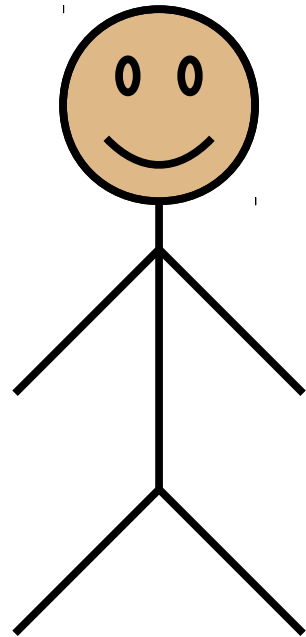
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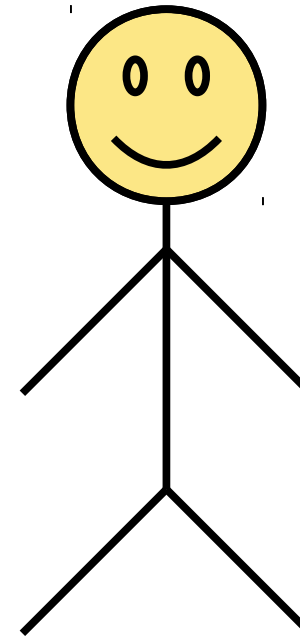
Proof Writer (You)

$k = 68$

Neither Picks

$n = 137$

Reader Picks



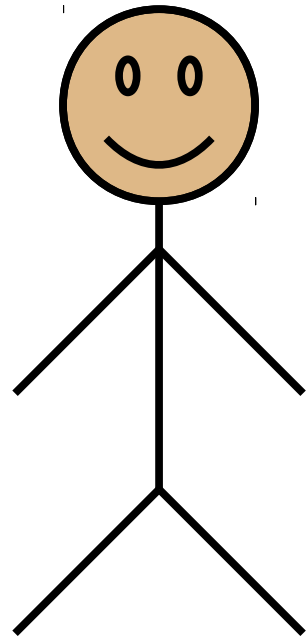
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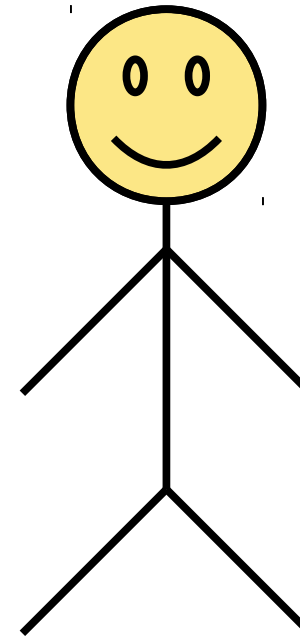
Proof Writer (You)

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Neither Picks

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Reader Picks



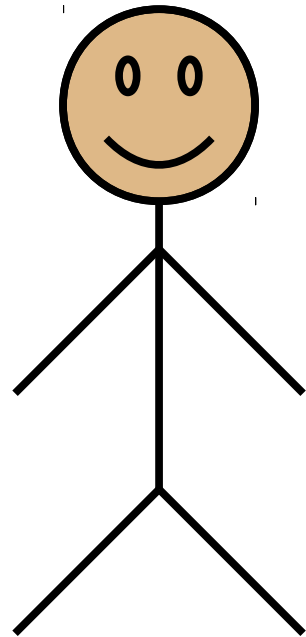
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Proof Writer (You)

$z = 34$

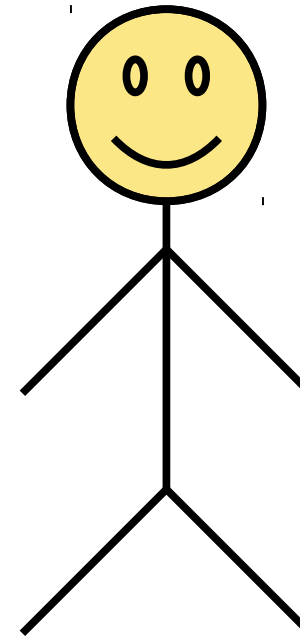
Writer Picks

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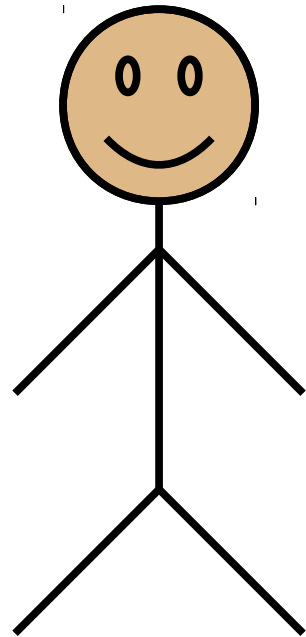
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Proof Writer (You)

$z = 34$

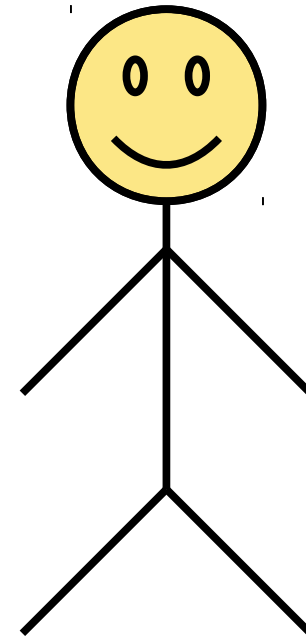
Writer Picks

$k = 68$

Neither Picks

$n = 137$

Reader Picks



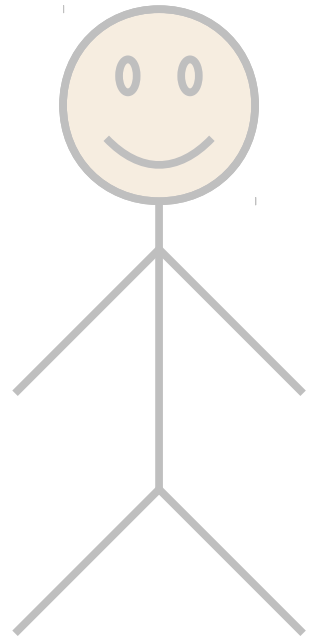
Proof Reader

Proofs as a Dialog

Pick an arbitrary odd integer n .

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.



Proof Writer (You)

$$z = 34$$

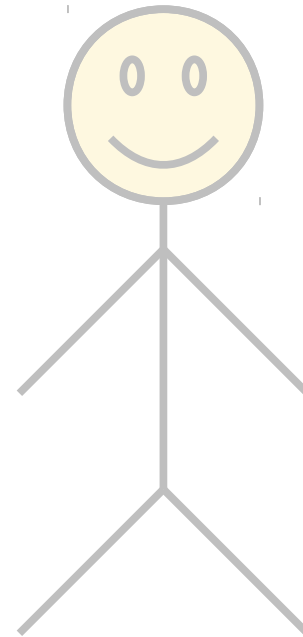
Writer Picks

$$k = 68$$

Neither Picks

$$n = 137$$

Reader Picks



Proof Reader

Each of these variables has a distinct, assigned value.

Since

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Now, let $z = k - 34$.

$$n = 137$$

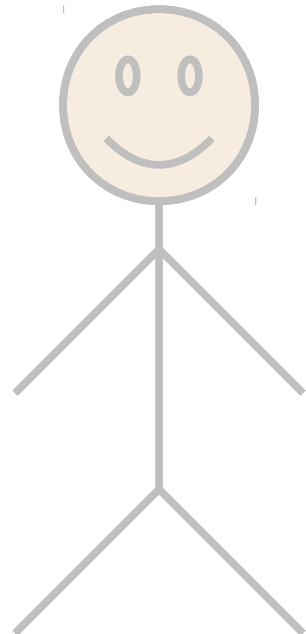
Reader Picks

$$k = 68$$

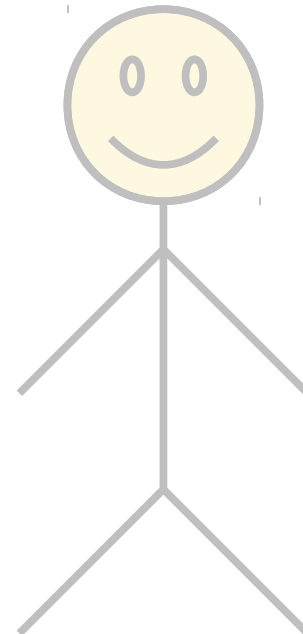
Neither Picks

$$z = 34$$

Writer Picks



Proof Writer (You)



Proof Reader

Who Owns What?

- The **reader** chooses and owns a value if you use wording like this:
 - Pick an arbitrary natural number n .
 - Consider some $n \in \mathbb{N}$.
 - Let n be a natural number.
 - Let n be an arbitrary $n \in \mathbb{N}$.
- The **writer** (you) chooses and owns a value if you use wording like this:
 - We choose $r = n + 1$.
 - Pick $s = n$.
- **Neither** of you chooses a value if you use wording like this:
 - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
 - Because n is odd, there must be some integer k where $n = 2k + 1$.

Proofwriting Rules We Learned Today

- Direct proof: The first two sentences of a proof of a theorem with “If...then...” form are (1) assume the “if” part, (2) announce you “want to show” the “then” part.
- To prove a universal, “pick an arbitrary.”
- To prove an existential, (1) announce a concrete value that works, then (2) justify that it works.
- Use formal definitions of terms.
- Write in complete sentences.
- Clearly introduce variable names using prescribed language. Don’t reuse/overlap variable names.
- Proof by Cases: You may divide a situation into all possible cases, and prove each one separately, to prove the whole. Give clear case labels, which act as assumptions.

Next Time

- ***Indirect Proofs***
 - How do you prove something without actually proving it?
- ***Mathematical Implications***
 - What exactly does “if P , then Q ” mean?
- ***Proof by Contrapositive***
 - A helpful technique for proving implications.
- ***Proof by Contradiction***
 - Proving something is true by showing it can't be false.